SIMULATION OF SOME PROCESSES OF DEVELOPMENT

R. Galas, H. Kwaśnicka, W. Kwaśnicki
Institut of Technical Cybernetics
Technical University of Wroclaw
Wroclaw, Poland

We consider successive generations of elements performing the same function in specific environment. We assume that elements are differently preferred with regard to their types, are reproduced with respect to these preferences and that sometimes the types of elements are modified during reproduction. We give the appropriate stochastic model describing the variations of the generations state resulting from such reproduction. We are of opinion that properties of this model fit well to those observed in some real life processes of development of evolutionary type. Several simulation examples are presented to uphold this thesis.

1. INTRODUCTION

We consider a population of elements of several types acting in some environment. All elements do essentially the same job - meet the same environment requirement, though they can accomplish it differently and be differently preferred by environment. We assume the specific environment in which quality index assigned to element and reflecting this preference is random variable with probability distribution the same for all elements of the same kind, acting in the same period of time. We will call the environment the more homogeneous the less is a variance of quality index. The environment has a limited capacity. Elements are introduced to environment, act there some time and are removed. The difference between current capacity and number of elements acting forms a demand for new elements.

We assume a discrete time and consider a process illustrated on fig. 1. In each period there exists a population which occupy the environment. It consists of that elements which "survived" from previous population and of new generation of elements supplied by reproduction process. The reproduction is a random process with tendency to meet the environment demand and to increase the number of elements with high quality on expense of elements with low quality. The elements of particular types are reproduced separately and the variance of reproduction ratio diminishes with growing number of elements reproduced. There exists a peak ratio of reproduction for each type which can not be exceeded and as a result the upper limit of elements number in next generation determining the extent to which the demand of environment can be fulfilled. When the peak ratios of reproduction are not involved, elements are reproduced in such a fashion that the expected number of all new elements equals the demand and expected value of reproduction ratio for particular type is proportional to its quality. There exists slight probability that element can be modified during reproduction into another type, sometimes with better quality, then those hitherto represented in the population. As a result of such a reproduction, generation state, defined as elements distribution between particular types, changes in time, what is follo-
wed by adequate changes in the population state.

In next section we propose a mathematical model describing the process of variation of the generation state in successive time periods. Using Waddington terms [1] it describes the behaviour of progressive system having the homeorheic property. We intend to point that discussed process exhibits a far going analogy with observed, real life development processes. In particular, there occurs the steady growth of the average generation quality, combined with cyclic changes of the generation state, as a successive types of higher and higher quality are introduced and gain domination in it.

There is quite a lot of papers, mainly biologically orientated, which deals with mathematical models of similiarly defined populations. It begins probably with Fisher [2], of more recent the work of Eigen [3], Karlin [4], Tuffner-Denker [5] should be mentioned in this context. The main innovation introduced in our model seems to be a vector space of types with quality function defined on it. It allowed the modification process to be adequately included into model.

One can meet remarks, e.g. [6] that other then biological development processes of evolutionary type can be considered in similar fashion. We tried to use neutral terms, and perhaps the reader can share our opinion that above description reflect to some extent, e.g. a process of competition on specific market.

2. MODEL

Let the type of element be determined by n parameters: $d_1, \ldots, d_n$ assuming integer values. Then each type $d^j$ can be represented as a point: $(d_1^j, \ldots, d_n^j)$ in n-dimensional discrete parameters space $D^n$:

$$D^n = \{d \in \mathbb{N}^n: x \in D^n \iff x \in \mathbb{N}^n \}$$

where $x = d, i = 1, \ldots, n$. (1)

Let the norm:

$$\|d^i - d^j\| = \sum_{k=1}^{n} |d_k^i - d_k^j|$$

(2)

gives a distance between types $i$ and $j$. Let $T = 0, 1, 2, \ldots$ denotes discrete time generation number. Let $G^T$ denote the $T$-th generation of elements. The state of generation is given by a distribution function:

$$N(\text{d}, T) = N^T \_i$$

determines the number of $i$-type elements in $G^T$. As $D^n$ is the denumerable set of points, all types $\text{d}$ can be numbered successively: $d_1, d_2, d_3, \ldots$ and the generation state can be denoted by the vector:

$$S^T = [N^T_1, N^T_2, N^T_3, \ldots]^T$$

(4)

We assume there are given in advance:

- Quality function:

$$q: R^n \times T \rightarrow R_+$$

(5)

$$q(x, T) = q_i$$

for $x = d^i$ determine expected quality of $i$-type elements in $G^T$.

- Maximum reproduction function:

$$r: D^n \times T \rightarrow R_+$$

(6)

$$r(d^i, T) = r^i$$

determines the peak reproduction ratio of $i$-type elements in $G^T$.

- Modification function:

$$m: D^n \times D^n \times T \rightarrow R_+$$

(7)

$$m(d^i, d^j, T) = m_{ij}$$

determines the probability of modification from $i$ to $j$-type during reproduction of $G^T$, $m_{ij} = 1$.

- Demand function:

$$M: T \rightarrow D_+$$

(8)

$M(T) = M^T$ determines the demand of environment for new elements in period $T$.

- Probability distribution of quality evaluation:

$$f(x, T) = q_i$$

(9)

where $f_i$ is a random variable with distribution $f$ determining the quality index assigned to $i$-type elements in $G^T$.

- Probability distribution of reproduction:

$$g(N_i, E[N_i^T]) = q_i^T$$

(10)

where $q_i^T$ is a random variable with distribution $g$ determining the number of "progeny" of $i$-type elements in $G^T$.

To evaluate the state of next generation we determine successively:

1. Average quality of $i$-type:

$$\bar{q}_i = \frac{1}{N_i^T} \sum_{k=1}^{N_i^T} \bar{q}_i^k, \quad \text{for all } i$$

(11)

where $\bar{q}_i^k$ is a realization of random variable for $k$-th element of $i$-type, eq. (9).

2. Total reproduction ratio of $i$-type:

$$r_i = \min \{a_i, r_i^j, r_{mi}\}, \quad \text{for all } i$$

(12)

where $a_i^\ast \in R_+$ is a factor common for all types adjusting reproduction to the demand:

$$a_i^\ast = \min_{d} \left\{ |N_i^T - \Sigma x_i^T r_i^j(a)| \right\}$$

(13)
3. Reproduction ratio from i to j-type
\[ r^T = r^T_{ij} m^T_{ij} \quad \text{for all } i, j \quad (14) \]

4. Expected state of T+1 generation:
\[ E[S^{T+1}] = [r^T_{ij}] S^T \quad (15) \]
where \( r^T_{ij} \) is an infinite-dimensional matrix of reproductions ratios.

5. State of T+1 generation:
\[ S^{T+1} = [N^{T+1}_1, N^{T+1}_2, N^{T+1}_3, \ldots]^T \quad (16) \]
where \( N^{T+1}_i \) is a realization of random "progeny" variable \( n_i^T \) eq. (10).

The above model describes to some extent the process discussed in introduction. Simulation results, presented further on, have been obtained under some simplifying assumptions which are as follows:

- We generated random values for the reproduction only, using the Poisson Distribution to this purpose - then the described process can be considered as a multitype branching process[7]. As the matter of fact, there is no need to generate random values for those processes separately, as one complex distribution can be educed and used for simulation.

- We assumed the constant demand M and the constant peak reproduction ratio \( m \) the same for all types. In such situation we could simplify the determining of reproduction ratio using instead eqs. (11) and (12) the formulæ:
\[ r^T = [1 + (r_m - 1) \frac{M - N^T_0}{N}] q^T_1 / q^T_0 \quad (17) \]
where \( N^T_0 \) is a total number of elements in \( G^T \) and \( q^T_0 \) is an average quality of \( G^T \):
\[ q^T_0 = \frac{1}{N^T_0} \sum N^T_i q^T_i \quad (18) \]
It can be shown that if \( r^T = q^T_1 / q^T_0 \) the expected number of all elements \( G^T \) is the same for all generations - the Critical Galton-Watson Process, If \( N^T_0 \) does not exceed M considerable and \( 1 < r_m < 2 \), what just occurred, remaining factor acts to reduce a difference between \( M \) and \( N^T_0 \).

- We assumed the modification probability \( m_{ij} \) depends only on distance between i and j-type and is given as follows:
\[ m_{ij} = \begin{cases} 0 & \text{if } \|d^1 - d^2\| > 2 \\ \frac{m_0^2}{m^0} & \text{if } \|d^1 - d^2\| = 2 \\ m_0 & \text{if } \|d^1 - d^2\| = 1 \\ \frac{1}{m^0} \sum m_{ij} & \text{if } \|d^1 - d^2\| = 0 \end{cases} \quad (19) \]
where \( m_0 \ll 1 \).

The last thing to discuss here is a choice of quality function, which is a matter of importance for appropriate interpretation of simulation results. We chose a shape roughly analogous to this of an undercut, narrow ridge gradually elevating in direction not parallel to the axes. It is a form similar to the Waddington's chreods [1], the only difference being - we deal with maximization not minimization. There is similar motivation for this choice also. We can often describe the development process of some objects indicating parameters achieved in successive generations and resulting trajectory correlated with quality growth. Any time, any serious deviation from this trajectory gives a rapid drop of quality. A better solution is not easy to be found and generally requires a change of several parameters simultaneously, as parameters are usually well tuned up. The chosen metric of \( D^t \) reflects this property as it "rates higher" the change involving more parameters.

3. SIMULATION

3.1. Typical features of simulated processes.

The dynamics of the generation state is a result of two trends composed. The first trend arising from selective reproduction, tends to increase the number of elements of types with quality \( q_i \) higher then average quality \( q_0 \), and to reduce the number of others. This tendency is the stronger, the greater (the lesser) is a ratio \( q_i / q_0 \). As an average quality increases during such process, this trend acts to supersede all elements by temporarily the best ones - to attract the generation distribution to the point representing the timely best type. The second trend, arising from modifications gives elements of types deviated from this being reproduced - it acts to disperse the generation distribution around. This two trends taken together can give a state of quasi-equilibrium. In the parameters space, there exists a center represented by the type temporarily the best containing a majority of elements and certain surrounding of this center, composed of types with worse quality but because of modifications still containing some elements. When due to modifications the elements of new better type are introduced to generation, the center of distribution moves to the point representing this new type.

To illustrate the typical features of generation dynamics we use data obtained from simulation carried out under following assumptions:
\[ M=N^0=10^4, \quad m_0=1.5 \cdot 10^{-3}, \quad n=2 \]
\[ N^0(15,15)=10^4 \quad \text{and zero elsewhere.} \]
\[ q(d_1+d_2)=\exp\{-10^{-2}[d_1+d_2+(d_2-d_1)^2]\} \quad (20) \]
Fig. 2. Parameter space, quality function and $G^{110}$ and $G^{250}$ states.

The state of 110-th and 250-th generations is shown on fig. 2. The generations are so distant, that they are composed of elements of entirely distinct types. The 110-th generation is in transition stage, the center of distribution just moves from point (8,8) to point (7,7). The 250-th generation is in quasi equilibrium stage, most of elements is concentrated in the point, (4,4) the better type has not yet been found. On the axes, the distributions of parameters for each generation have been indicated. The first moments of those distributions are the coordinates of generation center. Note, that logarithmic scale for the number of elements has been applied.

On the same figure the contour lines of the quality function are indicated. As can be seen, the function has such a shape that a change of single parameter of dominating type gives a type with worse quality. To get the better type it is necessary to modify two parameters in proper direction. In quasi-equilibrium state practically all elements are of dominating type and this type quality is the average quality of generation. As the assumed probability of double modification is $(1.5)^2 \cdot 10^{-5}$, there are two ways to get it and the number of elements is $10^6$, the successful double modification can be expected one a 45 generations. From comparison of this number with that obtained for double modifications follows that the main source of successful modifications is not the dominating type but the types belonging to its surrounding.

The "history" of generation is drawn on fig. 3. Successively types (7,7), (6,6), (5,5), (4,4) gains and lose domination in the generation. Most of the time there is only one type dominating. All the time, exists a "background" consisting of elements of types from surrounding of generation center. They have usually worse quality than those which dominate. Their number is a small fraction of total number of elements present in generation.

Fig. 3. Elements of particular types in successive generations.

The average quality obtained in successive generations is drawn on fig. 4.

Fig. 4. Average quality and variance in successive generations.
It can be seen that periods of rapid growth of quality are correlated with transition stages of generation. On the same figure the variance of generation is traced. The lowest level of variance is observed in quasi-equilibrium state. The simulation results presented in this paper are for deterministic quality evaluation and for random reproduction with Poisson distribution. The question can arise, to what extent the random factors impact the model behaviour. It appears there exists a strong deterministic component, especially when more numerous types or the best types are involved. Fig. 5 gives for successive generations the number of elements of two distinct types obtained in four different realizations. The results are for: \( M = 10^3 \), \( m_0 = 0 \). Initial state:

\[
q_1^T = q_1^T + 0.15 \sin \left( \frac{3}{2} T + \varphi \right),
\]

3 deterministic evaluation and deterministic reproduction,

4 deterministic evaluation and random reproduction with Poisson Distribution.

Numbers of elements for 3-rd and 5-th types are drawn. Irrespective of small size of generation - 100 elements in initial state - what increase the impact of random factors, realizations seems to be quite well tuned.

3.2. Impact of parameters on development rate

We consider the development rate as given by a slope of the curve representing the average quality of successive generation. To illustrate the influence of particular parameters of the model discussed on the development rate, we use data obtained from simulation, carried out with following quality function:

\[
q \left( d_1, d_2 \right) = \left( 1 - 2 \cdot 10^3 \left( d_1 + d_2 \right)^2 \left[ \frac{\Delta a}{\Delta e} \left( d_2 - d_1 \right) \right] \right)^{\frac{1}{2}}
\]

where: \( \Delta a = a_5 / a_1 \) - asymmetry coefficient giving a ratio of ellipse axes.

As this function is less steep then that given by eq. (20) the stages of transition and quasi-equivalence are less distinct.

Fig. 6. gives the average quality obtained for generation size \( N_0^1 \) equal: \( 10^4, 10^5, 10^6 \). The results are for: \( N = N_0^1 \), \( m_0 = 0.01 \), \( a = 2 \), \( S : N \left( 30, 30 \right) = N_0^1 \) and zero elsewhere.

![Fig. 5.](image)

![Fig. 6.](image)
Fig. 7. gives the average quality obtained for probability of modifications \( m \) equal: \( 10^{-1}, 10^{-2}, 10^{-3} \). The results are for: \( M=N^0=10^4, a=4, S^0: N^0(30,30)=10^4 \) and zero elsewhere. The greater is the probability of modifications the more rapid development is obtained.

Fig. 7. Impact of probability of modification on average quality

For \( m_0=10^{-4} \), after 120 generations dominates the best type possible: \( d(0,0)- \) steady state. Due to high rate of still occurring modifications the \( q^T \) is only 0.94 of maximal quality \( q(0,0)=1 \). This cut in quality can be considered as a price of rapid development.

Fig. 8. gives the average qualities obtained for different asymmetry coefficients \( a \) equal: 1, 4, 16. Results are for: \( M=N^0=10^4, m_0=10^{-2}, S^0: N^0(30,30)=10^4 \) and zero elsewhere. The greater is the asymmetry of axes the more difficult the development is.

Fig. 8. Impact of asymmetry of quality function on average quality.

Fig. 9. gives the map of quality functions with trajectories of the generation center for different starting points drawn in. We assumed that whole population is initially of one type respectively: \( d(11,11), d(13,7), d(15,4), d(15,0) \). The results are for: \( M=N^0=10^4, a=4 \).

Fig. 9. Generation center trajectories for different starting points.

Fig. 10 gives average qualities obtained from different starting points. It appears that the lowest rate of development is obtained when the center of generation proceeds along the ridge of quality function the same time trajectories tends to proceed along this ridge. As the shape of quality function reflects environmental preferences, the situation when generation finds itself outside the ridge can be interpreted as a result of a rapid change of environment.

Fig. 10. Impact of initial state on average quality.
3.3. Some special cases

Figs. 11 and 12 illustrate the generation behaviour when the quality function has a local maximum. The results are for different $m_0$ equal: $10^{-3}$, $10^{-2}$, $10^{-3}$, and $M=500=10^4$, $S_0$: $N(12, 12)=10^4$ and zero elsewhere. Quality function is given on fig. 11. The trajectory of generation for $m_0=10^{-2}$ has been drawn, marks the center position for every twenty generations.

![Fig. 11. Passage through local maximum.](image)

Fig. 11. Passage through local maximum. 

Fig. 12. illustrate the ratio of development with respect to $m_0$. The higher the modification probability is, the easier the generation pass through the local maximum. The price of more rapid development is a lower quality in stationary state - compare curves 2 and 3.

![Fig. 12. Passage through local maximum, probability of modification influence on average quality.](image)

Fig. 12. Passage through local maximum, probability of modification influence on average quality.

![Fig. 13. Passage through bifurcation point.](image)

Fig. 13. Passage through bifurcation point.

It is obvious that an assumption that probability of modification depends only on distance between elements is over simplified with respect to real life processes of development. Promising modification should be made "outside" of existing types and especially "at the forefront" of developing generation. It seems also in some sense easier to get a product uniting two existing features then a product with quite new feature - parameter value.
The last case to be shown here is case of "pursuit". We compare two generations on different level of development, developing separately along the same ridge. The curves 1 and 4 on fig. 15 show the average quality obtained by elements of this two generation. Results are for: $M=N^0=10^5$, $m_0=1.5 \times 10^3$, $q(d_1, d_2)$ is given by eq. (20), $S(4,4)=700$, $N(5,5)=9000$, $N(6,6)=100000$ and zero elsewhere for the more advanced and: $S^0$: $N(7,7)=100$, $N(8,8)=90000$.

Fig. 15. Pursuit strategies for less advanced generation.

3.4. Forecasting

A forecasting seems to be one of more promising fields of model applications. It can be carried out under following assumptions:

- a specific environment should be suitably delimited, i.e. expected preferences are the same for elements of the same types - it does not matter that only the best type dominates, and its demand defined.
- a quality function - preferences - does not vary in time or this variations are defined.
- a reproduction process regards the preferences in a fashion assumed.
- a sufficient interval of generation history is known to the extent which make possible to determine the number of all elements involved in the environment demand fulfilling in some previous generations.

It is not necessary to know the parameters of types and quality function shape.

The idea of forecasting is to estimate the relative qualities of particular types knowing the relative ratios of their reproduction in previous generations and to use these qualities to simulate the development assuming none new type will appear. Having sufficient set of such realizations we can determine the prognosis as an average of such realizations as well as it variance. Assuming the quality and time in which the new better type will appear, we can determine the range of prognosis e.g. as a time in which the number of this new type elements will be less than 10% of whole generation number.

To illustrate the problem of generation state forecasting we use a data obtained from simulation carried out under following assumptions: $N=10^5$, $m=10^3$, $q(d_1, d_2)=\exp(0.003(d_1+d_2)^2+16(d_2-d_4)^2)$ (22)

The initial state is characterised in table 1.
Table 1. Number and quality of elements of particular types in $S^0$.

<table>
<thead>
<tr>
<th>Type number</th>
<th>Quality type</th>
<th>Number</th>
<th>Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0.383</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.444</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>238</td>
<td>0.405</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>154</td>
<td>0.312</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.220</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>0.383</td>
<td>16</td>
</tr>
<tr>
<td>7</td>
<td>77</td>
<td>0.405</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>2054</td>
<td>0.379</td>
<td>18</td>
</tr>
<tr>
<td>9</td>
<td>3293</td>
<td>0.323</td>
<td>19</td>
</tr>
<tr>
<td>10</td>
<td>54</td>
<td>0.249</td>
<td>20</td>
</tr>
</tbody>
</table>

Fig. 16. gives the number of elements for some of this types, the new types which appeared as early as in 7-th generation have been also included.

Fig. 16. Elements number for some types in exemplary realization.

We "forecasted" this exemplary realization assuming that only information we have is a number and quality of elements in initial state and that none new type appears. Fig. 17. make possible to compare the exemplary realization, deterministic prognosis and one of random prognosis realization for some types.

The prognosis seems to be quite reasonable though more then 1 000 elements of better types existed in 30-generation.

Figs. 18 and 19. yield some information concerning random prognosis convergence. For: $N=1\ 000$, $N_s=100$, initial state:

- $N_0^0 = 40$, $N_0^0 = 30$, $N_0^0 = 20$, $N_0^0 = 8$, $N_0^0 = 2$,
- $q_t^0 = 1.0$, $q_t^0 = 1.1$, $q_t^0 = 1.2$, $q_t^0 = 1.3$, $q_t^0 = 1.4$,

some realizations have been computed. Four of this realization as well as average and deterministic prognosis are drawn for 3-rd and 5-th type respectively.
4. CONCLUSIONS

The model discussed in this paper is entirely abstract, simulation results have been obtained under quite arbitrary assumptions. The same time, the process the model describes manifests quite a lot of properties which are characteristic for some of real life development processes.

We believe there are e.g.
- development itself is a matter of preferences expressed in some environment,
- there are two trends composed: to supersede all elements by temporarily the best ones and to find still better elements,
- irrespectively of significant influence of random factors the growth curves we meet in development processes are very similar - logistic e.g. (fig. 5.),
- one can trace usually a posteriori - a trajectory of development in parameter space, there is a constant growth of quality along this trajectory,
- there are typical stages of development, the stages of transition when new better elements gain domination and the stages of quasi-equilibrium when the new better types of elements are mostly found (fig. 2.),
- there exist always elements of types "deviated" from this temporarily the best, successful modifications are usually modifications of this very deviated types,
- the more elements of deviated types exist and the more significant are this deviations the more rapid is a development process (fig. 12.),
- the price of development is a cut in current average quality of population,
- a sudden change in environmental preferences can release a rapid development figs. 9 and 10.,
- some directions of development can be missed and it is not easily reversible process fig. 13.,
- most of successful innovations are found as a result of recombination of existing types fig. 14.,
- a contact of isolated populations results usually in rapid development (fig. 15.).

We hope, the model presented will prove its usability as a forecasting tool and perhaps it can be of some use when the development control means are considered. As for a present, we try to check the model using real life data. The main difficulty is that accessible data are usually collected rather for specific type in different environments, then for different types in specific environment.

REFERENCES